ON A CERTAIN TYPE OF COMMUTATORS OF OPERATORS

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ABSTRACT

Let H be a separable infinite-dimensional Hilbert space and let C be a normal operator and G a compact operator on H. It is proved that the following four conditions are equivalent.

1. $C + G$ is a commutator $AB-BA$ with self-adjoint A.
2. There exists an infinite orthonormal sequence e_j in H such that $|\Sigma_{j=1}^{n} (Ce_j, e_j)|$ is bounded.
3. C is not of the form $C_1 \oplus C_2$ where C_1 has finite dimensional domain and C_2 satisfies $\inf \{ |(C_2 x, x)| : \|x\| = 1 \} > 0$.
4. 0 is in the convex hull of the set of limit points of sp C.

1. Introduction. Let H be a separable infinite-dimensional Hilbert Space. The class of commutators $AB-BA$ where A is a (bounded linear) hermitian operator and B is an arbitrary (bounded linear) operator on H will be denoted, as in [2], by X_H. The symbol Y_H denoting in [2] the subclass of commutators of X_H for which B is of the form iD where D is hermitian, will also be used here and will have the same meaning.

The class Y_H which is identical with the class of hermitian operators in X_H, was studied by H. Radjavi in [3]. His main result can be stated as follows:

(*) Each of the following three conditions is necessary and sufficient for a hermitian operator C to belong to Y_H.

a) There exists an infinite orthonormal sequence e_j in H such that

$$\left| \sum_{j=1}^{n} (Ce_j, e_j) \right| \text{ is bounded.}$$

b) C is not of the form $C_1 \oplus C_2$ where C_1 has finite dimensional domain and C_2 satisfies the condition

$$\inf_{\|x\|=1} \left|(C_2 x, x)\right| > 0.$$

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c) 0 is in the convex hull of the set of limit points of sp C.

We recall that a number z is a limit point of the spectrum of a normal operator C if either z is a point of accumulation of the spectrum in the topological sense or z is an eigenvalue of C with infinite multiplicity.

Necessary conditions and sufficient conditions for an arbitrary (not necessarily hermitian) operator to belong to X_H were established in [2]. Here are two propositions proved in [2].

i) Every compact operator on H is in X_H

ii) If C is a normal operator on H whose spectrum contains n limit points $z_1, z_2 \cdots z_n$ satisfying

$$m_1 z_1 + m_2 z_2 + \cdots + m_n z_n = 0$$

for a set $(m_1 m_2 \cdots m_n)$ of rational positive numbers, then $C \in X_H$.

It was shown also in [2] that conditions a) and b) in (*) are necessary for an arbitrary operator to be in X_H and that a normal operator C in X_H has property c).

We formulate now the main result of this paper:

THEOREM 1. *Let N be a normal operator and G be a compact operator on H. The operator $M = N + G$ is in X_H if and only if 0 is in the convex hull of the set of limit points of sp N.*

This theorem is a generalization of (i) and (ii). We show now that property (b) is a consequence of (a) for arbitrary operators C. In fact, suppose that C satisfies (a) and that (b) does not hold. Then by the Toeplitz-Hausdorff theorem that the numerical range of an operator is convex there exist a real number θ and two operators C_1 and C_2 such that

$$\mathrm{Re}(e^{i\theta}C) = \mathrm{Re}(e^{i\theta}C_1) \oplus \mathrm{Re}(e^{i\theta}C_2)$$

$\mathrm{Re}(e^{i\theta}C_1)$ has a finite dimensional domain and that

$$\inf_{\|x\|=1} \mathrm{Re}(e^{i\theta}C_2 x, x) = \inf_{\|x\|=1} (\mathrm{Re}(e^{i\theta}C_2)x, x) > 0$$

where $\mathrm{Re}\, A = \frac{1}{2}(A + A^*)$. On the other hand we obtain from (a) that the sequence

$$\sum_{j=1}^{n} ((\mathrm{Re}(e^{i\theta}C))e_j, e_j) = \mathrm{Re} \sum_{j=1}^{n} (e^{i\theta}Ce_j, e_j)$$

is bounded. This contradicts the equivalence of (a) and (b) in the case of the hermitian operator $\mathrm{Re}(e^{i\theta}C)$ (which follows from (*)).

Let us note that the fact that (b) is a consequence of (a) for arbitrary operators C can also be proved directly by a method similar to that used in the proof of corollary 2 in [3].

Since, as it was shown in [2], (b) and (c) are equivalent for normal operators one has, as a conclusion, the following generalization of Theorem (*).

THEOREM 2. *Let C be a normal operator and G be a compact operator on H. $C + G$ is in X_H if and only if C satisfies the three equivalent conditions (a), (b) and (c).*

2. Proof of Theorem 1. Suppose that M is in X_H and that 0 is not in the convex hull of the set of limit points of $\operatorname{sp} N$. Therefore, N is of the form $N_1 \oplus N_2$ where N_1 is defined on a finite dimensional domain and

$$\inf_{\|x\|=1} \left|(N_2 x, x)\right| > 0$$

It follows that there exists a real number θ such that the hermitian operator $C = \operatorname{Re}(e^{i\theta}N)$ does not satisfy (b). Hence C has not property (c). Since the set o limit points of $\operatorname{sp} C$ is also the set of limit points of $\operatorname{Re}(e^{i\theta}M) = C + \operatorname{Re}(e^{i\theta}G)$ (see [4] p. 367) it follows that (c) does not hold for $\operatorname{Re}(e^{i\theta}M)$. On the other hand, since $e^{i\theta}M$ is in X_H one obtains easily that $\operatorname{Re}(e^{i\theta}M)$ is in Y_H. This contradicts Theorem (*).

For the proof of the remaining part of the theorem suppose that 0 is in the convex hull of the set of limit points of $\operatorname{sp} N$. Therefore, there exist three (not necessarily distinct) limit points z_1, z_2 and z_3 in $\operatorname{sp} N$ satisfying.

$$m_1 z_1 + m_2 z_2 + m_3 z_3 = 0 \tag{1}$$

for three positive numbers m_1, m_2 and m_3 with $m_1 + m_2 + m_3 = 1$. Then by Lemma 3 in [2] N is unitary similar to an operator N' on

$$K = H \oplus H \oplus H \oplus H \oplus H \oplus H$$

of the form $P' \oplus Q'$ with

$$P' = \begin{bmatrix} N_1 + z_1 I & 0 & 0 \\ 0 & N_2 + z_2 I & 0 \\ 0 & 0 & N_3 + z_3 I \end{bmatrix}$$

and

$$Q' = \begin{bmatrix} N_4 + z_1 I & 0 & 0 \\ 0 & N_5 + z_2 I & 0 \\ 0 & 0 & N_6 + z_3 I \end{bmatrix}$$

where I is the identity operator on H and N_i is a normal operator having 0 as a limit point of its spectrum for $i = 1,2,3,4,5,6$. M is therefore unitary similar to an operator M' of the form $N' + G'$, where G' is a compact operator on K.

Let U_i $i = 1,2,3,4,5,6$, be any six unitary operators on H. Let

$$P'' = \begin{bmatrix} U_1^*N_1U_1 + z_1I & 0 & 0 \\ 0 & U_2^*N_2U_2 + z_2I & 0 \\ 0 & 0 & U_3^*N_3U_3 + z_3I \end{bmatrix}$$

and

$$Q'' = \begin{bmatrix} U_4^*N_4U_4 + z_1I & 0 & 0 \\ 0 & U_5^*N_5U_5 + z_2I & 0 \\ 0 & 0 & U_6^*N_6U_6 + z_3I \end{bmatrix}$$

It is easily seen that N' is unitary similar to $N'' = P'' \oplus Q''$. There exists, therefore, a compact operator G'' on K such that M' is unitary similar to $M'' = N'' + G''$.

The equality $m_1 + m_2 + m_3 = 1$ implies the existence of a three by three (complex) numerical unitary matrix $(b_{ij})_{i,j=1}^3$ with

$$b_{i3} = \sqrt{m_i} \qquad i = 1,2,3. \tag{2}$$

Let

$$U = (b_{ij}I)_{i,j=1}^3$$

$$V = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}$$

$P''' = U^*P''U$ and $Q''' = V^*U^*Q''UV$. Since U and V are unitary operators on $H \oplus H \oplus H$, M'' is unitary similar to an operator M''' of the form $(P''' \oplus Q''') + G'''$ where G''' is compact. Moreover, M''' has a matricial representation of the form

$$\begin{bmatrix} P''' + G_{11} & G_{12} \\ G_{21} & Q''' + G_{22} \end{bmatrix}$$

where G_{ij}, $i,j = 1,2$ are compact operators on $H \oplus H \oplus H$. Let

$$Z = \begin{bmatrix} \frac{1}{\sqrt{2}} I_{H\oplus H\oplus H} & \frac{1}{\sqrt{2}} I_{H\oplus H\oplus H} \\ -\frac{1}{\sqrt{2}} I_{H\oplus H\oplus H} & \frac{1}{\sqrt{2}} I_{H\oplus H\oplus H} \end{bmatrix}$$

where $I_{H\oplus H\oplus H}$ denotes the identity operator on $H\oplus H\oplus H$.

Z is obviously a unitary operator on K and an easy computation shows that

$$M^{(4)} = Z^*M'''Z = \begin{pmatrix} \frac{1}{2}(P''' + Q''' + F_{11}) & \frac{1}{2}(P''' - Q''' + F_{12}) \\ \frac{1}{2}(P''' - Q''' + F_{21}) & \frac{1}{2}(P''' + Q''' + F_{22}) \end{pmatrix}$$

where F_{ij} are compact operators.

By Theorem 8 in [2] $M^{(4)}$ is in X_K whenever $P''' + Q''' + F_{11}$ and $P''' + Q''' + F_{22}$ are in $X_{H\oplus H\oplus H}$. Moreover, using (1) and (2) one obtains by computation that $P''' + Q''' + F_{11}$ and $P''' + Q''' + F_{22}$ are both of the form

$$S = \begin{pmatrix} A & B & E_1 + G_1 \\ C & D & E_2 + G_2 \\ E_3 + G_3 & E_4 + G_4 & E_5 + G_5 \end{pmatrix}$$

where G_i are compact and E_i are linear combinations of $U_j^*N_jU_j$ $i = 1, 2, 3, 4, 5$. Since, by our constructions, M is unitary similar to $M^{(4)}$ and U_j are arbitrary it suffices to prove the next lemma to complete the proof of the theorem.

LEMMA 1. *Suppose that $N_j, j = 1,2,3,4,5,6$ are six normal operators such that 0 is a limit point of* sp N_j. *Then there exist six unitary operators U_j such that S is in $X_{H\oplus H\oplus H}$ for any four operators A,B,C,D, fire compact operators G_i and fire operators $E_i = \sum_{j=1}^{6} a_{ij}U_j^*N_jU_j$ where a_{ij} are complex numbers.*

PROOF OF LEMMA 1. We begin by citing a known lemma:

LEMMA 2. *Let J be a normal operator on H having 0 as a limit point of its spectrum. If $\varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_n > \cdots$ is a sequence of positive numbers converging to 0 then there exists a sequence R_k, $k = 1,2,3,\cdots$ of mutually orthogonal infinite dimensional subspaces of H such that*

1) $$H = \sum_{k=1}^{\infty} \oplus R_k$$

2) R_k *reduces* J

and

3) $$\|J/R_k\| < \varepsilon_k \text{ for } k = 2,3,4\cdots$$

(See [1] p. 701–702).

It follows easily from this lemma that H can be written in the form

$$H = \sum_{k=3}^{\infty} \oplus M_k$$

such that the following conditions are satisfied:

I) M_k are mutually orthogonal infinite dimensional subspaces of H.

II) There exist 6 unitary operators U_j on H with $U_1 = 1$ such that M_k are reducing subspaces for $U_j^* N_j U_j$, $j = 1,2,3,4,5,6$.

III) $\| U_j^* N_j U_j / M_k \| < (12(k!))^{-1}$ for $j = 1,2,3,4,5 6$ and $k = 4,5,6 \cdots$.

Let $c > \sup_{i,j}\{|a_{ij}|\}$ and $E_i' = (1/c)E_i$ for $i = 1,2,3,4,5$.

Then

II$_1$) M_k reduces E_i' for $k = 3,4,5 \cdots$ and $i = 1,2,3,4,5$ and

III$_1$) $\| E_i' / M_k \| < \sup_{i,j}\{|a_{ij}|\}/2c(k!) \leqq (2(k!))^{-1}$ for $k = 4,5,6 \cdots$, and $i = 1,2,3,4,5$.

Let $G_i' = (1/c)G_i$ for $i = 1,2,3,4,5$ Since G_i are compact, it follows, by Lemma 4.1 in [1], that

$$M_k = L_k \oplus P_k \qquad k = 4,5,6 \cdots$$

where L_k and P_k are orthogonal infinite dimensional subspaces of M_k and

$$\| G_i' x \| < \frac{1}{2(k!)} \| x \| \tag{3}$$

$$\| G_i'^* x \| < \frac{1}{2(k!)} \| x \|$$

for $i = 1,2,3,4,5$ and for each $x \neq 0$ in L_k.

Define

$$H_3 = M_3 \oplus \sum_{k=4}^{\infty} \oplus P_k$$

and

$$H_k = L_k \text{ for } k = 4,5,6 \cdots$$

It follows that

$$H = \sum_{k=3}^{\infty} \oplus H_k$$

and that inequalities (3) and

$$\| E_i' x \| < \frac{1}{2(k!)} \| x \| \tag{4}$$

$$\| E_i'^* x \| < \frac{1}{2(k!)} \| x \|$$

hold for $i = 1,2,3,4,5$ and each $x \neq 0$ in H_k, $k = 4,5,6\cdots$.
Now, let P_k be the projection of H on H_k for $k = 3,4,5,6\cdots$ Define

$$Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Q_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_k \end{bmatrix} \text{ for } k = 3,4,5,6\cdots$$

Then Q_k are orthogonal projections of $H \oplus H \oplus H$ for $k = 1,2\cdots$, and we have

$$H \oplus H \oplus H = \sum_{k=1}^{\infty} \oplus Q_k(H \oplus H \oplus H),$$

$$Q_1(H \oplus H \oplus H) = H \oplus 0 \oplus 0,$$

and

$$Q_2(H \oplus H \oplus H) = 0 \oplus H \oplus 0.$$

Let

$$Z_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and for $k = 3,4,5\cdots$ let Z_k be a linear transformation from $H \oplus H \oplus H$ to $H \oplus 0 \oplus 0$ mapping $Q_k\ (H \oplus H \oplus H) = 0 \oplus 0 \oplus H_k$ isometrically on $H \oplus 0 \oplus 0$ and annihilating $H \oplus H \oplus (H \ominus H_k)$. Then, by Lemma 3.1 in [1] $S' = (1/c)\,S$ is unitary similar to an infinite operator matrix $T = (T_{kl})_{k,l=1}^{\infty}$ on $(H \oplus 0 \oplus 0) \oplus (H \oplus 0 \oplus 0) \oplus \cdots$ where $T_{kl} = Z_k S Z_l^*$ for $k, l = 1,2,3\cdots$.

It is easily seen that for $k = 3,4,5,\cdots$, Z_k are of the form

$$Z_k = \begin{bmatrix} 0 & 0 & V_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where V_k is an operator on H mapping isometrically H_k on H and annihilating $H \ominus H_k$. The computation of T_{kl} is therefore easy and it yields:

$$T_{kl} = \begin{bmatrix} (E'_k + G'_k)\, V_l^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad \begin{matrix} k = 1,2 \\ l = 3,4,5,6\cdots \end{matrix}$$

$$T_{kl} = \begin{bmatrix} V_k(E'_{l+2} + G'_{l+2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad \begin{matrix} k = 3,4,5\cdots \\ l = 1,2 \end{matrix}$$

and

$$T_{kl} = \begin{bmatrix} V_k(E'_5 + G'_5)\, V_l^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad \begin{matrix} k = 3,4,5\cdots \\ l = 3,4,5\cdots \end{matrix}$$

Let $(x;0;0)$ be any vector in $H \oplus 0 \oplus 0$. Using (3) and (4) one obtains

$$\| T_{kl}(x;0;0) \| = \| (E'_k + G'_k) V_l^* x \| \leqq \frac{1}{l!} \| V_l^* x \| = \frac{1}{l!} \| x \| = \frac{1}{l!} \| (x;0;0) \|$$

for $k = 1,2$ and $l = 4,5,6\cdots$;

$$\| T_{kl}^*(x;0;0) \| = \| (E'^*_{l+2} + G'^*_{l+2})\, V_k^* x \| \leqq \frac{1}{k!} \| V_k^* x \| = \frac{1}{k!} \| x \|$$

$$= \frac{1}{k!} \| (x;0;0) \|$$

for $k = 4,5,6\cdots$ and $l = 1,2$;

$$\| T_{kl}(x;0;0) \| = \| V_k(E'_5 + G'_5)\, V_l^* x \| \leqq \| (E'_5 + G'_5)\, V_l^* x \| \leqq \frac{1}{l!} \| x \|$$

$$= \frac{1}{l!} \| (x;0;0) \| \text{ for } k = 3,4,5,6\cdots \text{ and } l = 4,5,6,\cdots;$$

and

$$\| T_{kl}^*(x;0;0) \| = \| V_l(E'^*_5 + G'^*_5)\, V_k^* x \| \leqq \| (E'^*_5 + G'^*_5)\, V_k^* x \| \leqq \frac{1}{k!} \| x \|$$

$$= \frac{1}{k!} \| (x;0;0) \|$$

for $k = 4,5,6\cdots$ and $l = 3,4,5\cdots$.

Hence $\| T_{kl} \| \leqq \min \left(\frac{1}{k!}, \frac{1}{l!} \right)$ whenever $\max(k, l) > 3$. Then by Theorem 3 in [2] $T \in X_{(H \oplus 0 \oplus 0) \oplus (H \oplus 0 \oplus 0) \oplus + \cdots}$. Since S' is unitary similar to T and $S = cS'$, it follows that $S \in X_{H \oplus H \oplus H}$. This completes the proofs of Lemma 1 and Theorem 1.

References

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